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Gravitational Instability and the Nebular Hypothesis

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XI. *Gravitational Instability and the Nebular Hypothesis.*

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Introduction.

§ 1. A CONSIDERATION of the processes of cosmogony demands an extensive knowledge of the behaviour of rotating astronomical matter. What knowledge we have is based upon the researches of MACLAURIN, JACOBI, POINCARÉ, and DARWIN. These researches refer solely to matter which is perfectly homogeneous and incompressible, although it is, of course, known that the primordial astronomical matter must be far from homogeneous and probably highly compressible as well. The question of how far we are justified in attributing to real matter the behaviour which is found to be true for incompressible and homogeneous matter is obviously one of great importance.

§ 2. There are *à priori* reasons for expecting that there will be wide differences between the two cases. Consider first a sphere of homogeneous incompressible matter devoid of rotation. This will be stable if every small displacement increases (or, at least, does not decrease) its potential energy. The sphere has a number of independent possible small displacements which can be measured by the number of harmonics which can be represented on its surface. The spherical configuration is known to be stable because it can be shown that every one of these displacements increases the potential energy.

Contrast this case with the corresponding one in which the matter is compressible. The number of possible small displacements in this latter case is measured by the sum of the numbers of harmonics which can be represented *on all the spherical surfaces inside the sphere.* Let R be the radius of the outer surface; let r, r', r'', ... be the radii of all the spheres which can be drawn inside this outer sphere, and let r_n, r'_n, r''_n ... R_n be the number of independent harmonics which can be represented on these spheres. To prove that the sphere is stable it is now necessary to prove that every one of the $r_n+r'_n+r''_n+ \ldots R_n$ possible displacements increases the potential energy. If we argue by analogy from the case of an incompressible sphere we are, in effect, merely considering R_n of these displacements and neglecting the much greater number $r_n+r'_n+r''_n+ \ldots$. Furthermore, in these neglected displacements, the nature of the displacement is essentially different from that in the

R_n displacements, so that there appears to be no justification at all for an argument from analogy.

In each of the neglected displacements, the change in the potential energy will consist of two terms. There will be a change in the elastic energy of the compressible material, and this can be easily shown to involve an increase in the potential energy. There will, in addition, be a change in the gravitational energy, and this can be shown* to involve invariably a *decrease* in the energy. If W, E, G denote the total, the elastic, and the gravitational potential energies,

$$\delta W = \delta E + \delta G,$$

in which δG is invariably negative. The condition for stability is that for every possible displacement δE shall be numerically greater than δG.

It might naturally be thought that by considering a system in which the matter was, so to speak, very gravitational or very little elastic we could have δE small or δG great, and so should have instability of the spherical configuration. But it must be remembered that the gravitation and the elasticity of the matter are not independently at our disposal. The action of the gravitational forces tends to concentrate the matter and so involves that the elasticity becomes large in the equilibrium configuration. If we consider a system in which the elasticity is artificially kept small, as, for instance, by adding an additional repulsive field of force to annul, or partially annul, the gravitational field, we can easily construct systems for which a spherical configuration is unstable,† but, short of this, it appears to be a general law that the elastic and gravitational agencies must march together in such a way that δE is always numerically greater than δG,‡ so that every natural spherical system is stable.

The nearest approach in nature to the artificial repulsive field imagined above is found in the influence of rotation. This influence may be represented by the superposition of the usual repulsive field of centrifugal force of potential $-\frac{1}{2}w^2(x^2+y^2)$. The field is not spherical, and so the figures of equilibrium obtained under its influence cannot be spherical. But it can be regarded as made up of a spherical part $-\frac{2}{3}w^2r^2$ and a superposed harmonic disturbance $\frac{2}{3}w^2P_2r^2$. The first term is certainly a spherical repulsive field, and will, of course, tend to annul the concentrating influence of gravitation. The problem which requires study is that of how far, or in what circumstances, the presence of rotation can disturb the otherwise general law that δE is always greater than δG.

The problem is one of enormous complexity and great generality. It will hardly be expected that the present paper will contain anything approaching a general

* *Cf.* J. H. JEANS, "The Stability of a Spherical Nebula," 'Phil. Trans.,' A, vol. 199, p. 1.

† *Cf.* J. H. JEANS, "On the Vibrations and Stability of a Gravitating Planet," 'Phil. Trans.,' A, vol. 201, p. 157.

‡ *Cf.* below, §§ 11, 22.

solution, and it may as well be stated at once that it does not. All I have been able to do is to grope after general principles by solving a problem here and a problem there as seemed needful to illuminate a possible path towards a general theory, and the present paper is confined to a very few of the special problems I have considered, but I have selected those which seemed to have most bearing on the general question in hand.

Medium in which the Pressure is a Function of the Density.

§ 3. In the most general astronomical medium the pressure is, of course, not a function of the density. The relation between pressure and density varies from point to point, partly on account of inequalities of temperature and partly on account of variations of chemical constitution. But no general theory can be expected to apply to the most general heterogeneous mass of matter possible, and before any general theory can be deduced we must have material from which to deduce it.

§ 4. The simple system from which we shall start will be a system in which the matter is homogeneous as regards its properties, so that at all points the pressure and density will be connected by the same relation. It will be seen later (§ 15) how it is possible, in at least one important respect, to escape from this limitation.

For the present we assume the pressure and density to be connected by the relation

$$p = f(\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at every point. We take the centre of gravity of the rotating mass to be the origin, and the axis of rotation to be the axis of z. The equations of equilibrium are

$$\frac{\partial p}{\partial x} = \rho \frac{\partial V}{\partial x} + w^2 \rho x,$$

$$\frac{\partial p}{\partial y} = \rho \frac{\partial V}{\partial y} + w^2 \rho y,$$

$$\frac{\partial p}{\partial z} = \rho \frac{\partial V}{\partial z},$$

in which V is the potential of the whole gravitational field of force. In virtue of relation (1), these equations have the common integral

$$\int \frac{dp}{\rho} = V + \tfrac{1}{2} w^2 (x^2 + y^2) + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\phi(\rho) = V + \tfrac{1}{2} w^2 \varpi^2 + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which ϖ^2 stands for $x^2 + y^2$, and $\phi(\rho)$ for $\int \frac{dp}{\rho}$, which is by hypothesis a function of

ρ only. There is further the relation of Poisson,

$$\nabla^2 \mathrm{V} = -4\pi\rho, \quad \ldots \quad (4)$$

so that on operating on (3) with ∇^2 we obtain

$$\nabla^2 \phi(\rho) + 4\pi\rho = 2w^2, \quad \ldots \quad (5)$$

the differential equation which must be satisfied by ρ in any configuration of equilibrium under a rotation w.

§ 5. In general a solution of equation (5) will involve negative and zero values of ρ. In the physical problem ρ will be limited as to values, and this limitation will determine the physical boundary of the rotating mass.

Let V_m denote the gravitational potential at any point in space of the finite mass determined in this way. We have found a configuration of equilibrium under a potential V, the potential of the mass is V_m, so that for equilibrium we require an additional field of potential $\mathrm{V}-\mathrm{V}_m$. We can say that the configuration found will be a true configuration of equilibrium under an external field of force of potential V_0 such that

$$\mathrm{V} = \mathrm{V}_0 + \mathrm{V}_m. \quad \ldots \quad (6)$$

And, inasmuch as $\nabla^2 \mathrm{V}_m = -4\pi\rho = \nabla^2 \mathrm{V}$, it is clear that $\nabla^2 \mathrm{V}_0 = 0$, so that the external field has poles only at the origin or at infinity. The condition that any solution shall lead to a configuration of equilibrium for a mass rotating free from external influence is, of course, $\mathrm{V}_0 = 0$.

§ 6. The simplest solution of equation (5) is obviously that in which ρ is a function of r only, but it is clear from (3) that this cannot give a free solution except when $w = 0$.

§ 7. The next simplest form of solution is that in which ρ is a function of z and ϖ only, and this can give a free solution. It includes, of course, as a particular case the system of Maclaurin spheroids. For this class of solutions every section at right angles to the axis of z is circular, and in any such section the lines of equal density are circles. The density at any point is of the form $\rho = f(\varpi, z)$.

Let O denote colatitude measured from Oz, and let ψ be azimuth measured from the plane of xz. The most general configuration which can be obtained by displacement of that just considered will have a law of density of the form

$$\rho = f_0(\varpi, z) + \sum_1^\infty f_s(\varpi, z) \cos s\psi.$$

It is easily seen that the separate cosine terms lead to independent displacements, and we shall for the moment only consider the displacement of the first order, for which the law of density is

$$\rho = f_0(\varpi, z) + f_1(\varpi, z) \cos \psi, \quad \ldots \quad (7)$$

where $f_1(\varpi, z)$ is a small quantity of the first order.

The boundary being a surface of constant pressure must also be a surface of constant density, say σ. The equation of the boundary is accordingly

$$f_0(\varpi, z) + f_1(\varpi, z)\cos\psi = \sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The whole mass inside this boundary may be regarded as composed of coaxial rings of matter as follows. Inside the figure of revolution $f_0(\varpi, z) = \sigma$, we suppose there to be a series of rings of density given by (7), while the surface inequality can be regarded as represented by the presence of rings on this figure of revolution of density proportional to $\cos\psi$.

On integration the potential V_m at any external point is seen to be of the form

$$V_m = \chi_0 + \chi_1 \cos\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where χ_0, χ_1 are functions of ϖ and z only.

Suppose now that the surface is so nearly spherical that spherical harmonic analysis may be used with reference to it, then, since V_m is a solution of LAPLACE's equation at all external points, and is also of the form (9), it must be of the form

$$V_m = \frac{A_0}{r} + \sum_1^\infty \left(\frac{A_s}{r^{s+1}}\right) P_s^1(\mu)\cos\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $\mu = \cos\theta$, and $P_s^1(\mu)$ is the usual tesseral harmonic $\frac{\partial}{\partial\theta} P_s(\mu)$. Moreover, since the centre of gravity of the mass is supposed to coincide with the origin, A_1 must vanish.

We have, from equation (3), if $V = V_m$,

$$V_m = \phi(\rho) - \tfrac{1}{2}w^2\varpi^2 - C$$
$$= \phi\{f_0(\varpi, z)\} + f_1(\varpi, z)\cos\psi\,\phi'\{f_0(\varpi, z)\} - \tfrac{1}{3}w^2r^2(1 - P_2) - C$$

at all internal points. Equating these two expressions, we must have *at the boundary*

$$\frac{A_0}{r} = \phi\{f_0(\varpi, z)\} - \tfrac{1}{2}w^2\varpi^2 - C,$$

$$f_1(\varpi, z)\,.\,\phi'\{f_0(\varpi, z)\} = \sum_2^\infty \left(\frac{A_s}{r^{s+1}}\right) P_s^1(\mu),$$

or, neglecting small quantities of the second order,

$$f_1(\varpi, z)\,\phi'(\sigma) = \sum_2^\infty \left(\frac{A_s}{r^{s+1}}\right) P_s^1(\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence either $f_1(\varpi, z)$ vanishes at the boundary or is of at least the *second* order of harmonics.

It follows that if there can be a configuration of equilibrium which differs from the configuration of revolution $\rho = f_0(\varpi, z)$, by a displacement proportional to the *first* harmonic, this configuration must be one in which $f_1(\varpi, z)$ vanishes at the boundary, so that the boundary must be a figure of revolution about the axis of rotation.

It now follows from (3) that V must be a function of ϖ and z only on the boundary, and hence also (since V is harmonic) at all external points. It follows that $\frac{\partial V}{\partial n}$, and hence also $\frac{\partial \rho}{\partial n}$, are functions of ϖ, z only at the boundary. Whence again, by equations (4) and (5), it follows that $\frac{\partial^2 V}{\partial n^2}$ and $\frac{\partial^2 \rho}{\partial n^2}$ are functions only of ϖ and z at the boundary. And, by successive differentiation of equations (4) and (5), it is seen that all the differential coefficients of V and ρ are functions only of ϖ and z at the boundary.

It can be seen from this* that the configuration must be one of revolution throughout. In other words, there can be no configuration of equilibrium which differs from the configuration of revolution by *first* harmonic terms only.

LAPLACE'S *Law*.

§ 8. I have not found that any progress worth recording can be made with the general relation $p = f(\rho)$, so that progress can only be hoped for by examining special cases.

The case that suggests itself as most important is that of the gas law $p = \kappa\rho$, satisfied in a perfectly gaseous nebula at uniform temperature. The difficulty is that such a nebula extends to infinity in all directions, and so cannot rotate as a rigid body. Or rather, when it is caused to rotate, it throws off its equatorial portions and the remainder rotates in the shape of an elongated spindle of infinite length. In this connection I have worked out the purely two-dimensional problem of a rotating gaseous cylinder of infinite length. The results are too long to be worth printing; it will, perhaps, suffice to record that the analysis bears out in full the conclusions arrived at in this paper.

The law which is most amenable to mathematical treatment is LAPLACE'S law

$$p = c(\rho^2 - \sigma^2),$$

or, as it is more convenient to write it,

$$p = \frac{2\pi}{\kappa^2}(\rho^2 - \sigma^2), \quad \ldots \ldots \quad (12)$$

in which c, ρ, κ, and σ are constants, σ being the value of the density at the free

* I have not succeeded in obtaining a rigorous proof of this. It might be objected that nothing in the above argument precludes first harmonic terms proportional to such a function as $e^{-\frac{1}{f(\varpi, z)}}$, where $f(\varpi, z) = 0$ is the equation of the boundary. The pure mathematician may not, although the astronomer will, be influenced by the consideration that such functions never occur in natural problems. If such a function did occur, it would involve an extremely fantastic relation between p and ρ.

surface. This law has the merit that the case of an incompressible fluid is covered by the special value $c = \infty$ or $\kappa = 0$, the density now having the value σ throughout.

There is the *à priori* objection to the law that its form precludes first harmonic displacements (*cf.* below, § 11). This objection would be fatal were it not that we have seen that first harmonic displacements are in any case of no importance. This being so, the objection falls to the ground, and I have thought it worth working out this law as far as possible.

§ 9. Using the relation (12), we have in place of the more general equations (3), (4), and (5), the particular equations

$$\frac{4\pi}{\kappa^2}\rho = V + \tfrac{1}{2}w^2\varpi^2 + C, \quad (13)$$

$$\nabla^2 V = -4\pi\rho, \quad (14)$$

$$\frac{4\pi}{\kappa^2}\nabla^2\rho + 4\pi\rho = 2w^2. \quad (15)$$

On putting

$$\rho - \frac{w^2}{2\pi} = \chi, \quad (16)$$

this last equation reduces to

$$(\nabla^2 + \kappa^2)\chi = 0. \quad (17)$$

No Rotation.

§ 10. When there is no rotation $w = 0$, $\chi = \rho$ and the equation becomes

$$(\nabla^2 + \kappa^2)\rho = 0.$$

The general solution is

$$\rho = \Sigma A_n r^{-1/2} J_{n+1/2}(\kappa r)\, S_n, \quad (18)$$

while the particular solution giving a spherical boundary is

$$\rho = A_0 r^{-1/2} J_{1/2}(\kappa r) = \frac{A_0}{\sqrt{\frac{1}{2}\pi\kappa}}\frac{\sin\kappa r}{r}, \quad (19)$$

the last being, of course, the well-known solution which occurs in LAPLACE'S theory of the figure of the earth. It will now be shown that this configuration is stable for all displacements.

§ 11. Let $r = a$ be the free surface corresponding to the simple solution (19). Consider an adjacent solution

$$\rho = A_0 r^{-1/2} J_{1/2}(\kappa r) + A_n r^{-1/2} J_{n+1/2}(\kappa r)\, S_n, \quad (20)$$

and let the corresponding free surface be

$$r = a + bS_n. \quad (21)$$

On substituting this value for r in (20), neglecting squares of b and equating corresponding harmonic terms, we obtain

$$\sigma = A_0 a^{-1/2} J_{1/2}(\kappa a) = \frac{A_0}{\sqrt{\frac{1}{2}\pi\kappa}} \frac{\sin \kappa a}{a}, \quad \ldots \quad (22)$$

$$A_n a^{-1/2} J_{n+1/2}(\kappa a) = -A_0 b \frac{d}{da}\{a^{-1/2} J_{1/2}(\kappa a)\} = \kappa A_0 b a^{-1/2} J_{3/2}(\kappa a), \quad \ldots \quad (23)$$

whence

$$b\sigma = A_n a^{-1/2} \frac{J_{n+1/2}(\kappa a)\, J_{1/2}(\kappa a)}{\kappa J_{3/2}(\kappa a)}. \quad \ldots \quad (24)$$

By integration, the value of V_m at a point on the sphere $r = a$ is found to be

$$V_m = \frac{4\pi}{a}\int_0^a A_0 r^{-1/2} J_{1/2}(\kappa r)\, r^2\, dr + \frac{4\pi}{(2n+1)\,a}\int_0^a \frac{A_n S_n}{a^n} r^{-1/2} J_{n+1/2}(\kappa r)\, r^{n+2}\, dr + \frac{4\pi a\sigma b}{2n+1} S_n$$

$$= \frac{4\pi A_0}{a\kappa} a^{3/2} J_{3/2}(\kappa a) + \frac{4\pi A_n}{(2n+1)\,a^{n+1}\kappa} a^{n+3/2} J_{n+3/2}(\kappa a)\, S_n$$

$$+ \frac{4\pi}{(2n+1)\,a^{n+1}} A_n a^{n+3/2} \frac{J_{n+1/2}(\kappa a)\, J_{1/2}(\kappa a)}{\kappa J_{3/2}(\kappa a)} S_n,$$

while the value of V, as given by equation (13), is

$$V = \frac{4\pi A_0}{\kappa^2} a^{-1/2} J_{1/2}(\kappa a) + \frac{4\pi A_n}{\kappa^2} a^{-1/2} J_{n+1/2}(\kappa a)\, S_n + \text{cons.}$$

If we put

$$V - V_m = v = v_0 + v_n S_n,$$

we obtain, after some reduction,

$$v_n = \frac{4\pi A_n a^{1/2}}{(2n+1)\,\kappa}\left\{J_{n-1/2}(\kappa a) - \frac{J_{n+1/2}(\kappa a)\, J_{1/2}(\kappa a)}{J_{3/2}(\kappa a)}\right\}. \quad \ldots \quad (25)$$

In general, this gives the value of A_n which determines the tide raised by a field of potential $v_n\left(\frac{r}{a}\right)^n S_n$ proportional to S_n. We notice that when $n = 1$, $v_n = 0$ or $A_n = \infty$ independently of the values of κ, a. This merely expresses the obvious fact that there can be no equilibrium at all so long as the fluid is acted on by a force proportional to a harmonic of the first order.

If it is possible for there to be a configuration of equilibrium when $v_n = 0$, other than that given by $A_n = 0$, this configuration will of course determine a point of bifurcation in the series of symmetrical configurations. The points of bifurcation are accordingly given by $v_n = 0$, or by

$$\frac{J_{n-1/2}(\kappa a)}{J_{n+1/2}(\kappa a)} = \frac{J_{1/2}(\kappa a)}{J_{1/2}(\kappa a)}. \quad \ldots \quad (26)$$

For brevity in printing, let us introduce the function u_n defined by

$$u_n = \frac{J_{n+1/2}(\kappa a)}{J_{n-1/2}(\kappa a)} = -\frac{d}{d(\kappa a)} \log\{(\kappa a)^{-n+1/2} J_{n-1/2}(\kappa a)\}. \quad \ldots \quad (27)$$

Near $\kappa a = 0$, $u_n = \dfrac{\kappa a}{2n+1}$; the value of u_1 is $\dfrac{1}{\kappa a} - \cot \kappa a$, and successive u's satisfy the difference-equation

$$u_{n+1} = \frac{2n+1}{\kappa a} - \frac{1}{u_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

With the help of these properties it is easy to draw approximate graphs of the curves $y = u_n$. Such a graph, for values of κa up to the first zero of u_1 ($\kappa a = 4{\cdot}49$) is represented in fig. 1, in which the vertical scale is $2\frac{1}{2}$ times the horizontal scale.

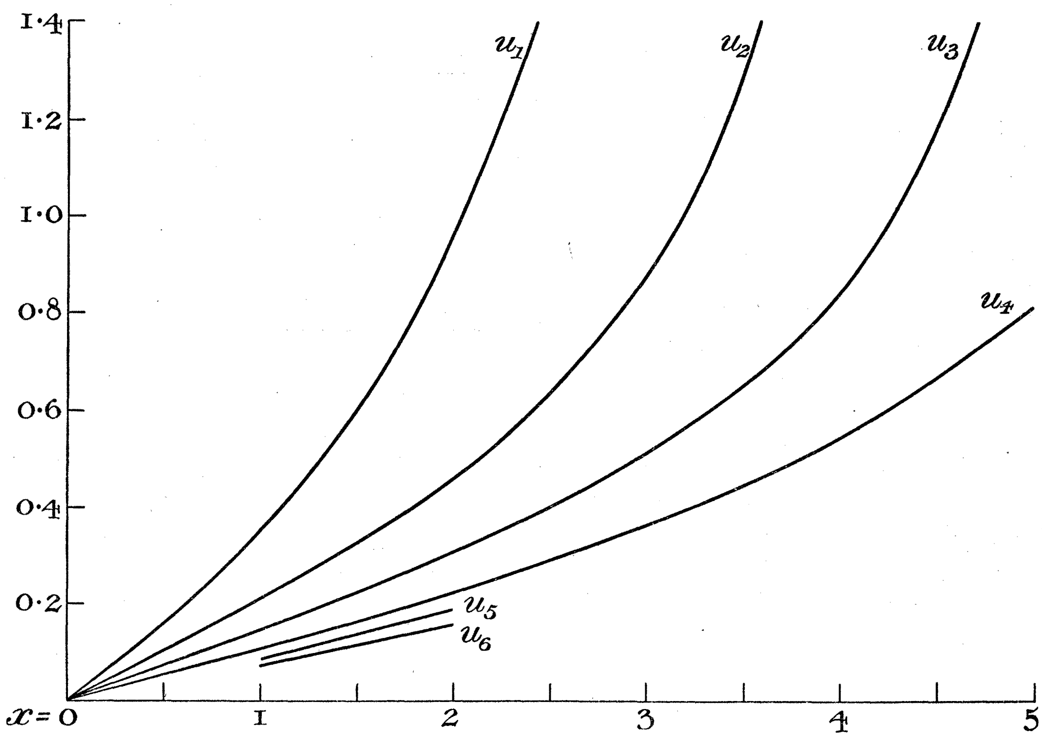

Fig. 1. Graphs of the functions u_n.

In terms of these functions, the points of bifurcation are given by $u_n = u_1$. It is at once evident that there is no root of this equation for values of κa less than π, and therefore (*cf.* equation (19)) no point of bifurcation at all so long as ρ is restricted to being always positive. It follows that the spherical configuration is stable for all displacements.

Small Rotation.

§ 12. When the fluid experiences a slight rotation w, the spherical configuration is of course slightly flattened. The appropriate solution of equation (17) is

$$\chi = \rho - \frac{w^2}{2\pi} = A_0 r^{-1/2} J_{1/2}(\kappa r) + A_2 r^{-1/2} J_{5/2}(\kappa r) P_2, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where P_2 is the second zonal harmonic about the axis of rotation as $\theta = 0$. Assuming the free surface to be

$$r = a + bP_2, \quad (30)$$

the equations analogous to (22), (23), and (24) are found to be

$$\sigma - \frac{w^2}{2\pi} = A_0 a^{-1/2} J_{1/2}(\kappa a), \quad (31)$$

$$A_2 a^{-1/2} J_{5/2}(\kappa a) = \kappa A_0 b a^{-1/2} J_{3/2}(\kappa a), \quad (32)$$

$$b\sigma = A_2 a^{-1/2} \frac{J_{5/2}(\kappa a) J_{1/2}(\kappa a)}{\left(1 - \frac{w^2}{2\pi\sigma}\right) J_{3/2}(\kappa a)}. \quad (33)$$

Let v be given by

$$v = V + \tfrac{1}{2} w^2 \varpi^2 - V_m = V - V_m + \tfrac{1}{3} w^2 r^2 (1 - P_2),$$

then, instead of equation (25), we have

$$v = \frac{4\pi A_2 a^{1/2}}{5\kappa} \left\{ J_{3/2}(\kappa a) - \frac{J_{5/2}(\kappa a) J_{1/2}(\kappa a)}{\left(1 - \frac{w^2}{2\pi\sigma}\right) J_{3/2}(\kappa a)} \right\} P_2, \quad (34)$$

in which constant terms are omitted, and the value is taken on the sphere $r = a$. For a configuration of equilibrium under no external field of force we must have $V = V_m$, and therefore v in equation (34) equal to $-\frac{1}{3} w^2 a^2 P_2$. Neglecting squares of w^2, and therefore omitting the factor $1 - \frac{w^2}{2\pi\sigma}$ in the denominator, the equation becomes

$$\tfrac{1}{3} w^2 a^2 = \frac{4\pi}{5\kappa} A_2 a^{1/2} J_{1/2}(\kappa a)(u_1 - u_2), \quad (35)$$

giving A_2 in terms of w^2, when w^2 is small. It will be readily verified that this equation is identical with that obtained by THOMSON and TAIT ('Nat. Phil.,' § 824, equation (14)).

§ 13. We next examine the solution

$$\chi = \rho - \frac{w^2}{2\pi} = A_0 r^{-1/2} J_{1/2}(\kappa r) + A_2 r^{-1/2} J_{5/2}(\kappa r) P_2 + A_n r^{-1/2} J_{n+1/2}(\kappa r) S_n, \quad (36)$$

which is appropriate to a mass of fluid having a rotation w given by equation (35), and acted on by a field of force of potential $v_n S_n$. By analysis exactly similar to that just given, we obtain at $r = a$

$$v_n = \frac{4\pi A_n a^{1/2}}{(2n+1)\kappa} \left\{ J_{n-1/2}(\kappa a) - \frac{J_{n+1/2}(\kappa a) J_{1/2}(\kappa a)}{\left(1 - \frac{w^2}{2\pi\sigma}\right) J_{3/2}(\kappa a)} \right\}. \quad (37)$$

This gives A_n for the general tide raised by the field $v_n S_n$. The condition for a point of bifurcation is $v_n = 0$, or

$$u_n = \left(1 - \frac{w^2}{2\pi\sigma}\right) u_1. \quad (38)$$

Thus the points of bifurcation, if any, are still determined by the intersections of the graphs in fig. 1, except that the graph of u_1 must be supposed decreased vertically in the ratio $1 - \frac{w^2}{2\pi\sigma}$ to 1.

We may, if we please, imagine that we start with very small rotation, and allow the rotation progressively to increase, this increase being accompanied in imagination by a greater and greater flattening of the graph of u_1.

It is clear that under all circumstances the curve which will first be intersected by the flattened graph of u_1 will be the graph of u_2. It is further clear that the requisite value of w^2 is least when $\kappa a = 0$, and progressively increases as κa increases, at any rate up to $\kappa a = \pi$.

This means that in the first place the circular vibration will invariably become unstable through a vibration proportional to a second harmonic, so that the first point of bifurcation reached will be one such that the spheroidal form gives place to an ellipsoidal form. If the rotation is so small that the problem may be treated as a statical one, there will be no question as to there being an actual exchange of stabilities at the point of bifurcation, for clearly v_n changes sign at this point. Thus for rotation greater than that at the point of bifurcation, the spheroidal form will be definitely unstable, and the ellipsoidal form definitely stable, at least until the next point of bifurcation is reached.

Our result shows, in the second place, that the masses which become ellipsoidal for the smallest values of w^2 are those for which κa is smallest. To put it briefly, the mass which is most unstable when it begins to rotate is the incompressible mass—a somewhat unexpected result.

For any value of κa, the value which w^2 must have for the spheroidal form to become unstable is (*cf.* equation (38))

$$1 - \frac{w^2}{2\pi\sigma} = \frac{u_2}{u_1}, \quad (39)$$

and when $\kappa a = 0$, the value of $u_2/u_1 = \frac{3}{5}$ (*cf.* § 4).

Thus our equations would make the spheroidal mass of incompressible fluid first become unstable when $\frac{w^2}{2\pi\sigma} = \cdot 400$, but these equations have only been obtained on the supposition that $\frac{w^2}{2\pi\sigma}$ is so small that its squares may be neglected, a supposition which is now seen *à posteriori* to be hardly admissible. Probably the results obtained are qualitatively true, but quantitatively unreliable. In point of fact the first point of bifurcation for an incompressible mass, instead of being given by $\frac{w^2}{2\pi\sigma} = \cdot 400$, is known to be given by the widely different value $\frac{w^2}{2\pi\sigma} = \cdot 1871$.

Our analysis has nevertheless proved rigorously the point which is really most important, namely, that there can be no point of bifurcation at all for quite small values of $\frac{w^2}{2\pi\sigma}$. At the same time, since the question of when and how a rotating mass first becomes unstable is one of considerable importance, I have attempted to obtain a more reliable investigation than the preceding. I have found that the accuracy is not greatly improved by taking the analysis as far as squares of $w^2/2\pi\sigma$, while the labour of working with a general power series would be appalling. I have, therefore, reluctantly been compelled to give up hopes of carrying the rigorous solution of the problem further in this direction, but have thought it worth while to examine the analogous problem for rotating cylindrical masses. All the essential physical features of the natural three-dimensional problem appear to be reproduced in the simpler cylindrical problem, so that it seems legitimate to hope that an argument by analogy may not lead to entirely erroneous result.

Cylindrical Masses in Rotation.

§ 14. The fundamental equations are, of course, the first two of the equations already written down in § 3. The third equation does not occur, since $\frac{\partial}{\partial z} = 0$. The equations have, as before, the integral (13) leading to the differential equation (15) for ρ.

The most general solution possible will be

$$\rho = \frac{w^2}{2\pi} + \sum_0^n \mathrm{A}_n \mathrm{J}_n(\kappa r) \cos(n\theta - \epsilon), \quad \ldots \quad (40)$$

in which r now stands for $\sqrt{(x^2+y^2)}$. No matter how great the rotation, there is always a special circular solution

$$\rho = \frac{w^2}{2\pi} + \mathrm{A}_0 \mathrm{J}_0(\kappa r), \quad \ldots \quad (41)$$

this being analogous to the spheroidal figures of equilibrium investigated in § 12.

Let us examine the deformed solution

$$\rho = \frac{w^2}{2\pi} + \mathrm{A}_0 \mathrm{J}_0(\kappa r) + \mathrm{A}_n \mathrm{J}_n(\kappa r) \cos n\theta, \quad \ldots \quad (42)$$

in which A_n is supposed small, but there are no restrictions on the value of $\frac{w^2}{2\pi}$. If the free surface $\rho = \sigma$ is supposed given by (*cf.* equations (21) and (30))

$$r = a + b \cos n\theta,$$

then, as in equations (31) and (32), a and b must satisfy

$$\sigma = \frac{w^2}{2\pi} + A_0 J_0(\kappa a). \qquad (43)$$

$$A_n J_n(\kappa a) = -A_0 b\kappa J'_0(\kappa a) = A_0 b\kappa J_1(\kappa a),$$

whence

$$b\sigma = A_n \frac{J_0(\kappa a) J_n(\kappa a)}{\kappa\left(1-\frac{w^2}{2\pi}\right) J_1(\kappa a)}. \qquad (44)$$

The potential of the mass, V_m, can be regarded as arising from a distribution of density ρ inside the cylinder $r = a$, together with a surface density $b\sigma \cos n\theta$ spread over the surface of the cylinder.

The first part of the potential, evaluated at R, Θ, is

$$C - \iint [\log\{r^2 + R^2 - 2rR\cos(\theta-\Theta)\}] \left[\frac{w^2}{2\pi} + A_0 J_0(\kappa r) + A_n J_n(\kappa r)\cos n\theta\right] r\,dr\,d\theta$$

$$= C - 2\iint \left[\log R - \sum_1^\infty \frac{r^s}{sR^s}\cos s(\theta-\Theta)\right]\left[\frac{w^2}{2\pi} + A_0 J_0(\kappa r) + A_n J_n(\kappa r)\cos n\theta\right] r\,dr\,d\theta$$

$$= A_n \int_0^a \frac{2\pi}{n}\frac{r^n}{R^n} J_n(\kappa r)\, r\,dr \cos n\Theta + \text{ terms independent of } \Theta$$

$$= \frac{2\pi}{\kappa n} A_n J_{n+1}(\kappa a)\frac{a^{n+1}}{R^n}\cos n\Theta + \text{ terms independent of } \Theta.$$

The potential of the surface distribution is

$$\frac{2\pi\sigma b a^{n+1}}{nR^n}\cos n\Theta,$$

so that, at $r = a$,

$$V_m = \left\{\frac{2\pi a}{\kappa n} A_n J_{n+1}(\kappa a) + \frac{2\pi b\sigma a}{n}\right\}\cos n\theta + \text{ terms independent of } \theta,$$

while, by equation (5),

$$V = \frac{4\pi}{\kappa^2}\left\{\frac{w^2}{2\pi} + A_0 J_0(\kappa r) + A_n J_n(\kappa r)\cos n\theta\right\} - \tfrac{1}{2}w^2 r^2.$$

If, as before, we express the tide-generating potential $V - V_m$ in the form $v_0 + v_n \cos n\theta$, we obtain for the value of v_n, at $R = a$,

$$v_n = A_n\left\{\frac{4\pi}{\kappa^2} J_n(\kappa a) - \frac{2\pi a}{\kappa n} J_{n+1}(\kappa a)\right\} - \frac{2\pi b\sigma a}{n}$$

$$= \frac{2\pi a}{\kappa n} A_n \left\{ J_{n-1}(\kappa a) - \frac{J_0(\kappa a) J_n(\kappa a)}{\left(1-\frac{w^2}{2\pi\sigma}\right) J_1(\kappa a)}\right\}. \qquad (45)$$

It will be seen that this equation is exactly analogous to the former equation (37), but with the important difference that the present equation is true for all values of w^2, without limit. The points of bifurcation are given by $v_n = 0$, or

$$u_{n-1/2}(\kappa a) = \left(1 - \frac{w^2}{2\pi\sigma}\right) u_{1/2}(\kappa a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

which again is exactly analogous to the former equation (38). The graphs of the functions $u_{1/2}$, $u_{3/2}$, ... will be found to lie as in fig. 2, and we may again imagine that

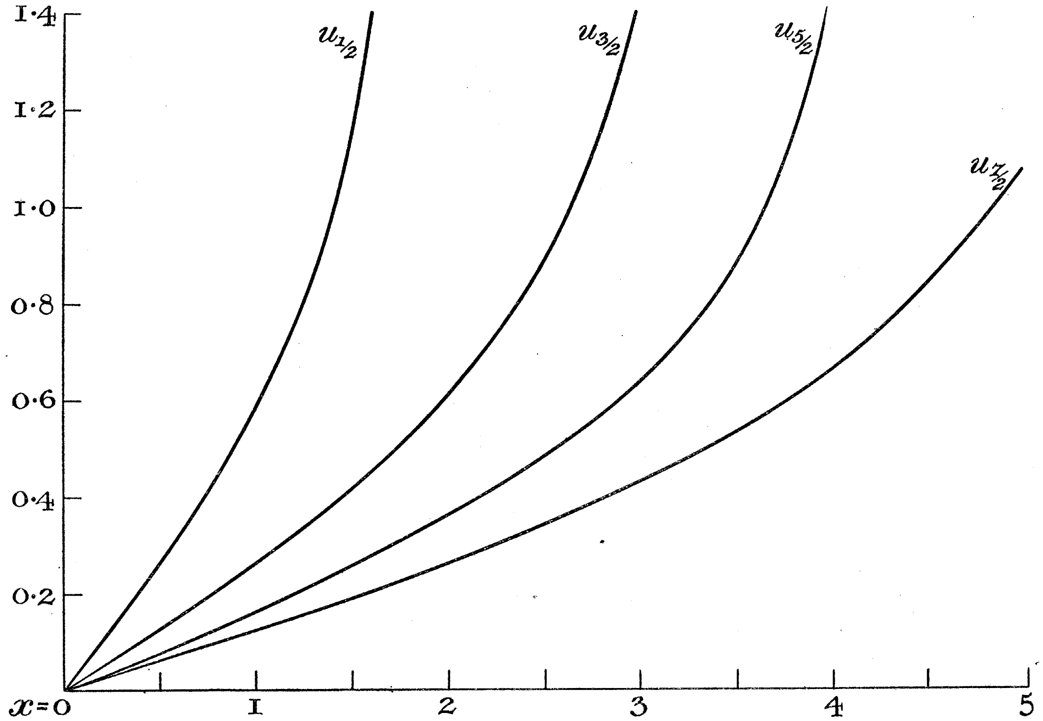

Fig. 2. Graphs of the functions $u_{n+1/2}$.

points of bifurcation are sought by flattening the curve $u_{1/2}$ until it intersects the other curves.

It is clear that, under all circumstances, the first curve to be intersected will be the curve $u_{3/2}$, corresponding to a displacement proportional to $\cos 2\theta$. Thus, as before, when the circular form becomes unstable, it gives place to a form of elliptic cross-section, which is stable. Moreover, the smaller κa is the lower the value of $w^2/2\pi\sigma$ for which the circular form becomes unstable.

These results are true without any regard to the value of w^2, so that they confirm the results stated, but not rigorously proved, in § 5. The numerical calculations which follow will make the matter clearer.

If $\bar{\rho}$ denotes the mean density of the rotating mass, the total mass per unit length is given by

$$\bar{\rho}\pi a^2 = 2\pi \int_0^a \rho r \, dr = 2\pi \left\{ \frac{w^2 a^2}{4\pi} + A_0 \frac{a_n J_1(\kappa a)}{\kappa} \right\},$$

giving, on substitution from equation (43),

$$\bar{\rho} = \frac{w^2}{2\pi} + \frac{2}{\kappa a}\frac{J_1(\kappa a)}{J_0(\kappa a)}\left(\sigma - \frac{w^2}{2\pi}\right) = \frac{w^2}{2\pi} + \left(1 + \frac{J_2(\kappa a)}{J_0(\kappa a)}\right)\left(\sigma - \frac{w^2}{2\pi}\right),$$

whence, for the ratio of $\bar{\rho}$ to σ, we have the general formula

$$\frac{\bar{\rho}}{\sigma} = 1 + \frac{J_2(\kappa a)}{J_0(\kappa a)}\left(1 - \frac{w^2}{2\pi\sigma}\right).$$

For the particular configuration which occurs at a point of bifurcation,

$$1 - \frac{w^2}{2\pi\sigma} = \frac{J_0(\kappa a)\, J_2(\kappa a)}{J_1^2(\kappa a)},$$

so that

$$\frac{\bar{\rho}}{\sigma} = 1 + \left\{\frac{J_2(\kappa a)}{J_1(\kappa a)}\right\}^2 = 1 + u^2_{3/2},$$

whence we obtain

$$\frac{w^2}{2\pi\bar{\rho}} = \frac{u_{1/2} - u_{3/2}}{u_{1/2}(1 + u^2_{3/2})}.$$

In the following table I have calculated the values of $w^2/2\pi\sigma$ and of $w^2/2\pi\bar{\rho}$ for which cylinders of different radii (a) and compressibility (κ) first become elliptical in cross section :—

κa.	$u_{1/2}$.	$u_{3/2}$.	$\frac{w^2}{2\pi\sigma}$.	$\frac{w^2}{2\pi\bar{\rho}}$.
0	0·0000	0·000	0·500	0·500
$\frac{1}{4}$	0·1261	0·063	0·503	0·500
$\frac{1}{2}$	0·2582	0·126	0·510	0·502
$\frac{3}{4}$	0·4040	0·192	0·525	0·506
1	0·5751	0·261	0·546	0·511
2	2·575	0·612	0·762	0·554
2·4048	∞	0·829	1·000	0·593
3	−1·304	1·433	2·099	0·687
3·8317	0·000	∞	∞	1·000

It will be seen that the general result is fully confirmed, that incompressible masses are the first to become unstable, and that the more compressible the mass is, the greater is the rotation required for it to depart from a symmetrical configuration.

Rotating nearly Spherical Mass with High Internal Temperature.

§ 15. We now leave the artificial two-dimensional problem and return to the real problem in three dimensions discussed in § 13.

The coefficient κ was there assumed to have the same value throughout the mass, as of course it would if the matter were homogeneous and of uniform temperature throughout. But to represent natural astronomical conditions there is no question that κ ought to increase on passing from the centre to the surface, thus representing a mass in which the temperature is highest inside and falls towards the surface.

We are in this way led to study the question of stability when κ is a function of r. It would be difficult to say precisely what function ought to be chosen if we were trying to represent natural conditions as faithfully as possible. It appears, however, that no continuous function will lead to equations which admit of integration. The only case which appears to be soluble is that in which the matter, before rotation, may be treated as if formed of a series of different layers, each being homogeneous and at a uniform temperature in itself, but the temperature varying from layer to layer. To represent this we take different values of κ in the different layers, κ being smallest in the interior.

There is no limit to the number of layers which can be treated analytically, but the assumption of a great number of layers naturally leads to highly complicated formulæ which are capable of conveying their meaning only after laborious numerical calculations. Both in order to obtain comprehensible results and to simplify the argument, the layers will, in what follows, be supposed to be only two in number. They may conveniently be referred to as the core and the crust. It will be found possible to generalize the results obtained so as to apply to any number of layers.

§ 16. We accordingly suppose that there is an interior core of radius a, in which the coefficient of compressibility has the uniform value κ, and that outside this is the crust of external radius c, in which the coefficient is κ'. It is again necessary to suppose the rotation to be so small that w^2 may be neglected.

As in § 3, the density ρ must satisfy

$$(\nabla^2+\kappa^2)\left(\rho-\frac{w^2}{2\pi}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

throughout the core, and the same equation with the appropriate value of κ throughout the crust. The most general solution of equation (47) is

$$\rho = \frac{w^2}{2\pi} + \sum_0^\infty \{A_n r^{-1/2} J_{n+1/2}(\kappa r) + B_n r^{-1/2} J_{-(n+1/2)}(\kappa r)\} S_n . \quad . \quad . \quad . \quad . \quad (48)$$

In the former problem all the terms in B_n could be omitted because ρ had to be finite at the origin. In discussing the solution for the crust these terms must be retained. The solution can, however, be put in a more concise form.

Let the constants A_n, B_n be replaced by new constants C_n, θ_n given by

$$A_n = C_n \cos\theta_n, \qquad B_n = C_n \sin\theta_n,$$

and let us introduce a function $J_{n+1/2}(x, \theta)$ defined by

$$J_{n+1/2}(x, \theta) = J_{n+1/2}(x)\cos\theta + J_{-(n+1/2)}(x)\sin\theta . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Then the solution (48) may be replaced by

$$\rho = \frac{w^2}{2\pi} + \sum_0^\infty C_n r^{-1/2} J_{n+1/2}(\kappa r, \theta_n) S_n, \quad . \quad . \quad . \quad (50)$$

which is formally analogous to (7).

The following properties of the function $J_{n+1/2}(x, \theta)$ may readily be verified, and will be required later :—

$$\frac{d}{dx}\{x^{n+1/2} J_{n+1/2}(x, \theta)\} = x^{n+1/2} J_{n-1/2}(x, -\theta), \quad . \quad . \quad . \quad (51)$$

$$\frac{d}{dx}\{x^{-(n+1/2)} J_{n+1/2}(x, \theta)\} = -x^{-(n+1/2)} J_{n+3/2}(x, -\theta), \quad . \quad . \quad . \quad (52)$$

$$J_{n+3/2}(x, \theta) + J_{n-1/2}(x, \theta) = \frac{2n+1}{x} J_{n+1/2}(x, -\theta). \quad . \quad . \quad . \quad (53)$$

There is a ready rule for writing down the values of these functions. In the first place, we have

$$J_{1/2}(x) = \frac{\sin x}{\sqrt{\tfrac{1}{2}\pi x}}, \qquad J_{-1/2}(x) = \frac{\cos x}{\sqrt{\tfrac{1}{2}\pi x}},$$

so that

$$J_{1/2}(x, \theta) = \frac{\sin(x+\theta)}{\sqrt{\tfrac{1}{2}\pi x}}.$$

Now let $\phi(x+\theta)$ be used to denote a general function made up of circular functions of $x+\theta$ and of algebraic functions of x. Then $J_{1/2}(x, \theta)$ is of the form $\phi(x+\theta)$, and any number of differentiations with respect to x, or of multiplications by powers of x, will still leave it in this form. It follows from (52) that $J_{3/2}(x, -\theta)$, $J_{5/2}(x, \theta)$, &c., will be of this form. Hence we have the general law

$$J_{n+1/2}(x, \theta) = \phi\{x + (-1)^n \theta\}, \quad . \quad . \quad . \quad (54)$$

in which the functional form of ϕ is at once given by

$$\phi(x) = J_{n+1/2}(x).$$

[For instance,

$$J_{5/2}(x) = \left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x,$$

so that

$$J_{5/2}(x, \theta) = \phi(x+\theta) = \left(\frac{3}{x^2} - 1\right)\sin(x+\theta) - \frac{3}{x}\cos(x+\theta).]$$

§ 17. We proceed to carry out analysis similar to that of § 13. Suppose that under a tide-generating potential $v_n S_n$, and a rotation w, the core assumes a configuration such that its boundary $r = a$ becomes deformed into

$$r = a + bS_n + \beta P_2, \quad . \quad . \quad . \quad (55)$$

while the outer surface becomes

$$r = c + d\mathrm{S}_n + \delta\mathrm{P}_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Let us suppose the densities in the two layers to be

$$\rho = \frac{w^2}{2\pi} + \mathrm{A}_0 r^{-1/2}\mathrm{J}_{1/2}(\kappa r) + \mathrm{A}_n r^{-1/2}\mathrm{J}_{n+1/2}(\kappa r)\,\mathrm{S}_n + \mathrm{A}_2 r^{-1/2}\mathrm{J}_{5/2}(\kappa r)\,\mathrm{P}_2 \quad . \quad . \quad . \quad (57)$$

in the core, and

$$\rho = \frac{w^2}{2\pi}\mathrm{C}_0 r^{-1/2}\mathrm{J}_{1/2}(\kappa r, \alpha) + \mathrm{C}_n r^{-1/2}\mathrm{J}_{n+1/2}(\kappa r, \beta)\,\mathrm{S}_n + \mathrm{C}_2 r^{-1/2}\mathrm{J}_{5/2}(\kappa r, \gamma)\,\mathrm{P}_2 \quad . \quad . \quad (58)$$

in the crust. The boundary between the two layers must clearly be an equipotential, and therefore a surface of constant pressure and density. Let σ, σ' be the densities at this boundary in the core and crust respectively.

On replacing r by $a + b\mathrm{S}_n + \beta\mathrm{P}_2$ in equations (57) and (58), the values of ρ must become σ and σ' respectively. This leads to the relations

$$\sigma - \frac{w^2}{2\pi} = \mathrm{A}_0 a^{-1/2}\,\mathrm{J}_{1/2}(\kappa a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\sigma' - \frac{w^2}{2\pi} = \mathrm{C}_0 a^{-1/2}\,\mathrm{J}_{1/2}(\kappa' a, \alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$b\left(\sigma - \frac{w^2}{2\pi}\right) = \mathrm{A}_n a^{-1/2}\,\frac{\mathrm{J}_{n+1/2}(\kappa a)\,\mathrm{J}_{1/2}(\kappa a)}{\kappa\mathrm{J}_{3/2}(\kappa a)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$b\left(\sigma' - \frac{w^2}{2\pi}\right) = \mathrm{C}_n a^{-1/2}\,\frac{\mathrm{J}_{n+1/2}(\kappa' a, \beta)\,\mathrm{J}_{1/2}(\kappa' a, \alpha)}{\kappa\mathrm{J}_{3/2}(\kappa' a, -\alpha)}. \quad . \quad . \quad . \quad . \quad . \quad (62)$$

From similar analysis applied to the outer boundary, if σ_0 is now the density at this boundary,

$$\sigma_0 - \frac{w^2}{2\pi} = \mathrm{C}_0 c^{-1/2}\,\mathrm{J}_{1/2}(\kappa' c, \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$d\left(\sigma_0 - \frac{w^2}{2\pi}\right) = \mathrm{C}_n c^{-1/2}\,\frac{\mathrm{J}_{n+1/2}(\kappa' c, \beta)\,\mathrm{J}_{1/2}(\kappa' c, \alpha)}{\kappa'\mathrm{J}_{3/2}(\kappa' c, -\alpha)}. \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Similar equations, of course, connect the coefficients which depend on the rotation.

The value of V_m at a point on the sphere $r = c$ can now be written down, as in § 11, and is found to be

$$\mathrm{V}_m = \frac{4\pi}{c}\left[\frac{w^2}{2\pi}\frac{c^3}{3} + \frac{\mathrm{C}_0}{\kappa'}\left\{c^{3/2}\mathrm{J}_{3/2}(\kappa' c, -\alpha) - a^{3/2}\mathrm{J}_{3/2}(\kappa' a, -\alpha)\right\} + \frac{\mathrm{A}_0}{\kappa}a^{3/2}\mathrm{J}_{3/2}(\kappa a)\right]$$
$$+ \frac{4\pi\mathrm{S}_n}{(2n+1)\,c^{n+1}}\,\mathrm{U}_n, \quad . \quad . \quad . \quad (65)$$

in which the rotational terms proportional to the second harmonic are omitted and U_n is given by

$$U_n = \frac{C_n}{\kappa'}\{c^{n+3/2} J_{n+3/2}(\kappa' c, -\beta) - a^{n+3/2} J_{n+3/2}(\kappa' a, -\beta)\} + \frac{A_n}{\kappa} a^{n+3/2} J_{n+3/2}(\kappa a) + d\sigma_0 c^{n+2} + b(\sigma - \sigma') a^{n+2}. \quad (66)$$

The value of V at $r = c$ is, from equation (13),

$$V = \frac{4\pi}{\kappa'^2}\left\{\frac{w^2}{2\pi} + C_0 c^{-1/2} J_{1/2}(\kappa c, \alpha) + C_n c^{-1/2} J_{n+1/2}(\kappa c, \beta) S_n\right\} - \tfrac{1}{2} w^2 (x^2 + y^2) + \text{cons.}, \quad (67)$$

whence, evaluating $V - V_m$ and picking out the coefficient of S_n, we find as the value of v_n at $r = c$,

$$v_n = \frac{4\pi}{\kappa'^2} C_n c^{-1/2} J_{n+1/2}(\kappa c, \beta) - \frac{4\pi}{(2n+1)c^{n+1}} U_n. \quad (68)$$

As before, the points of bifurcation, if any, are given by $v_n = 0$.

§ 18. It is now necessary to consider the boundary conditions which must be satisfied at the junction of the two layers. The condition of continuity of material, *i.e.*, that the inner surface of the crust shall coincide with the outer surface of the core, has been expressed in equation (55), b and β being the same for both core and crust. There is an equation of continuity of pressure expressed by

$$\frac{\sigma^2 - \sigma_{00}{}^2}{\kappa^2} = \frac{\sigma'^2 - \sigma_{00}{}^2}{\kappa'^2}, \quad (69)$$

which σ_{00} is now used to represent the density associated with zero pressure. Finally, there is a condition of continuity of normal force and this requires careful discussion.

Let M_1 denote the mass of matter actually forming the core and let V_1 denote its potential at any point outside the core. Let M_2 denote the mass which would replace the core if the solution (58) for the crust were extended to the centre and let V_2 denote the potential of this mass at any external point. It will be noticed that if solution (58) were extended to the origin, it would give an infinite density ρ at the origin and also an infinite value of V in virtue of equation (5). On the other hand, it is readily found, by direct integration, that V_2 the potential of the mass M_2, is finite at every point, including the origin. It follows that V can only be the potential of this imaginary arrangement of matter when it is acted on by certain external forces of which the potential becomes infinite at the origin. Let V_3 represent the potential of these forces. The value of V_3 is readily found, for it must satisfy $\nabla^2 V_3 = 0$, and must coincide with V or with $\frac{4\pi\rho}{\kappa^2}$ to within an additive constant at the origin. Thus V_3 is the limit of the right-hand side of equation (67) when $r = 0$,

r replacing c. This is found to be

$$\mathrm{V}_3 = \frac{4\pi}{\kappa'^2}\left[\frac{\mathrm{C}_0\,(\frac{1}{2}\pi)^{1/2}\sin\alpha}{r} + \frac{\mathrm{C}_n\,(\frac{1}{2}\pi)^{1/2}\,1\,.\,3\,.\,5\,\ldots\,(2n-1)\,(-1)^n\sin\beta}{r^{n+1}}\,\mathrm{S}_n\right].$$

The condition now to be satisfied is clearly that

$$\frac{\partial}{\partial r}\,\mathrm{V}_1 = \frac{\partial}{\partial r}\,(\mathrm{V}_2+\mathrm{V}_3)$$

at all points on the boundary $r = a+b\mathrm{S}_n$. This requires that $\mathrm{V}_1-\mathrm{V}_2-\mathrm{V}_3$ shall vanish, to within a constant, at all points outside this boundary, and therefore, in particular, at $r = c$. It will be readily seen that

$$\mathrm{V}_1-\mathrm{V}_2 = \mathrm{V}_m-(\mathrm{V}_m)_{\alpha=0}$$

while V_3 is exactly the value of the terms in V_m that involve α, when α is put equal to zero. The conditions sought are, therefore, simply that all the terms in α which occur in V_m shall vanish at every point of the sphere.

§ 19. We may now equate the coefficients of the separate harmonics, and obtain

$$\frac{\mathrm{C}_0}{\kappa'}\,\mathrm{J}_{3/2}(\kappa' a, -\alpha) = \frac{\mathrm{A}_0}{\kappa}\,\mathrm{J}_{3/2}(\kappa a)\,, \quad \ldots \quad (70)$$

$$\frac{\mathrm{C}_n}{\kappa'}\,\mathrm{J}_{n+3/2}(\kappa' a, -\beta) + b\left(\sigma' - \frac{w^2}{2\pi}\right)a^{1/2} = \frac{\mathrm{A}_n}{\kappa}\,\mathrm{J}_{n+3/2}(\kappa a) + b\left(\sigma - \frac{w^2}{2\pi}\right)a^{1/2}. \quad \ldots \quad (71)$$

On account of the simplifications made possible by these equations, equation (68) may be put in the form

$$v_n = \frac{4\pi}{\kappa'^2}\,\mathrm{C}_n c^{-1/2}\mathrm{J}_{n+1/2}(\kappa c, \beta) - \frac{4\pi}{(2n+1)\,c^{n+1}}\left\{\frac{\mathrm{C}_n}{\kappa'}\,c^{n+3/2}\mathrm{J}_{n+3/2}(\kappa' c, -\beta) + d\sigma_0 c^{n+2}\right\}$$

$$= \frac{4\pi c^{1/2}}{2n+1}\,\frac{\mathrm{C}_n}{\kappa'}\left\{\mathrm{J}_{n-1/2}(\kappa' c, -\beta) - \frac{\mathrm{J}_{n+1/2}(\kappa' c, \beta)\,\mathrm{J}_{1/2}(\kappa' c, \alpha)}{\left(1-\dfrac{w^2}{2\pi\sigma_0}\right)\mathrm{J}_{3/2}(\kappa' c, -\alpha)}\right\}. \quad \ldots \quad (72)$$

The elimination of A_0 and C_0 from (59), (60), and (70) gives

$$\left(\sigma' - \frac{w^2}{2\pi}\right)\frac{\mathrm{J}_{3/2}(\kappa' a, -\alpha)}{\kappa'\mathrm{J}_{1/2}(\kappa' a, \alpha)} = \left(\sigma - \frac{w^2}{2\pi}\right)\frac{\mathrm{J}_{3/2}(\kappa a)}{\kappa\mathrm{J}_{1/2}(\kappa a)}, \quad \ldots \quad (73)$$

while similarly the elimination of A_n and C_n from (61), (62), and (71) gives

$$\left(\sigma' - \frac{w^2}{2\pi}\right)\left\{1 + \frac{\mathrm{J}_{n+3/2}(\kappa' a, -\beta)\,\mathrm{J}_{3/2}(\kappa' a, -\alpha)}{\mathrm{J}_{n+1/2}(\kappa' a, \beta)\,\mathrm{J}_{1/2}(\kappa' a, \alpha)}\right\} = \left(\sigma - \frac{w^2}{2\pi}\right)\left\{1 + \frac{\mathrm{J}_{n+3/2}(\kappa a)\,\mathrm{J}_{3/2}(\kappa a)}{\mathrm{J}_{n+1/2}(\kappa a)\,\mathrm{J}_{1/2}(\kappa a)}\right\}. \quad . \quad (74)$$

For brevity we introduce a function $u_n(x, \theta)$, a generalisation of the u_n of § 11, the new function being defined by

$$u_n(x, \theta) = -\frac{\partial}{\partial x}\log\{x^{-n+1/2}J_{n-1/2}(x, \theta)\} = \frac{J_{n+1/2}(x, -\theta)}{J_{n-1/2}(x, \theta)}. \quad . \quad . \quad . \quad (74a)$$

Equations (73) and (74) now become

$$\frac{\sigma' - \frac{w^2}{2\pi}}{\kappa'} u_1(\kappa' a, \alpha) = \frac{\sigma - \frac{w^2}{2\pi}}{\kappa} u_1(\kappa\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

$$\left(\sigma' - \frac{w^2}{2\pi}\right)\{1 + u_1(\kappa' a, \alpha)\, u_{n+1}(\kappa' a, \beta)\} = \left(\sigma - \frac{w^2}{2\pi}\right)\{1 + u_1(\kappa a)\, u_{n+1}(\kappa a)\}, \quad . \quad (76)$$

while (72) becomes

$$v_n = \frac{4\pi c^{1/2}}{2n+1}\frac{C_n}{\kappa'} J_{n-1/2}(\kappa' c, -\beta)\left\{1 - \frac{1}{1 - \frac{w^2}{2\pi\sigma_0}}\frac{u_n(\kappa' c, -\beta)}{u_1(\kappa' c, \alpha)}\right\}, \quad . \quad . \quad . \quad . \quad (77)$$

so that points of bifurcation are given by

$$u_n(\kappa' c, -\beta) = \left(1 - \frac{w^2}{2\pi\sigma_0}\right) u_1(\kappa' c, \alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Again, if the rotation may be treated as small, there will invariably be a change of stability at these points of bifurcation, since v_n/C_n changes sign on passing through one of them.

§ 20. It is at once clear that the method can be extended to a mass consisting of any number of layers—the only difficulty occurs in the numerical computations at the end. At each boundary between two consecutive layers there will be equations of continuity precisely similar to (75) and (76), while the final value of v_n will be given by an equation exactly similar to (77), which it will be seen involves only quantities associated with the outer boundary.

The procedure in any particular case will be to start, so to speak, with the innermost core of the system. Equation (75) is linear in $\cos\alpha$ and $\sin\alpha$, so that $\tan\alpha$ is uniquely determined. Leaving out of account systems in which the density is, in any part, negative, this will be found to be adequate to determine α uniquely. Equation (76) now becomes a linear equation in $\cos\beta$ and $\sin\beta$, from which β can be determined uniquely. In this way, passing from layer to layer, we can determine the various values of α, β for the different layers. Finally, the α's and β's being known, equation (78) can be regarded either as an equation for w^2 or as an equation for c, *i.e.*, it can be regarded either as determining the highest rotation for which the symmetrical configuration is stable for a given value of c, or as determining the largest value of c for which the mass is stable under a given rotation. If the value of $\frac{w^2}{2\pi\sigma_0}$ obtained by the first method is not small, the result will be inaccurate; if the value for c obtained by the second method is so great that the density is in places negative, the result will be of no interest except as proving stability for smaller values of c.

§ 21. It may be well to take a general survey of the equations before giving special calculations. For simplicity we again consider two layers only, core and crust. From (75) and (76) it is clear that, when $a = 0$ or $\kappa = \kappa'$ (involving $\sigma = \sigma'$), the values of α and β vanish. Broadly speaking, the more distinct the core is from the crust, the larger α and β are. Equation (78), of course, differs only from the corresponding equation previously found, by the presence of the terms α and β. The effect of these terms is seen on noticing that, in the notation already used, $u_1(\kappa' c, \alpha)$ is of the form $\phi(\kappa' c - \alpha)$. Thus, to allow for the effect of the core on the term $u_1(\kappa' c, \alpha)$, we have to leave the algebraic part of the function unaltered, but to change all the trigonometrical arguments from $\kappa' c$ to $\kappa' c - \alpha$. Speaking very broadly, the general effect on the graph of u_1 (*cf.* fig. 1) is a compromise between leaving the graph unaltered and moving it bodily a distance α along the axis. Similar statements apply to the graph of u_n. Thus, while rotation as before is represented by flattening the graph of u_1 in fig. 1, the presence of a core is represented by a distortion of the graphs which may, with some truth, be thought of as bodily movements parallel to the axis. These bodily movements may cause new intersections between the graph u_1 and the other graphs, and the points of intersection will represent points of bifurcation at which the symmetrical configuration will become unstable.

No Rotation.

§ 22. The case that may properly be inspected first is that of no rotation. The equations reduce to

$$\frac{\sigma'}{\kappa'} u_1(\kappa' a, \alpha) = \frac{\sigma}{\kappa} u_1(\kappa a), \quad \ldots \ldots \quad (79)$$

$$\sigma' \{1 + u_1(\kappa' a, \alpha) u_{n+1}(\kappa' a, \beta)\} = \sigma \{1 + u_1(\kappa a) u_{n+1}(\kappa a)\}, \quad \ldots \quad (80)$$

and, the equation for points of bifurcation,

$$u_n(\kappa' c, -\beta) = u_1(\kappa' c, \alpha). \quad \ldots \ldots \quad (81)$$

When $n = 1$, it is seen that $\beta = -\alpha$ is a solution of (80), and must therefore (§ 20) be the only solution. To verify that $\beta = -\alpha$ is a solution, replace β by $-\alpha$ in (80) and it becomes

$$\sigma' \left\{1 + \frac{J_{5/2}(\kappa' a, \alpha)}{J_{1/2}(\kappa' a, \alpha)}\right\} = \sigma \left\{1 + \frac{J_{5/2}(\kappa a)}{J_{1/2}(\kappa a)}\right\}, \quad \ldots \ldots \quad (82)$$

which is seen to be identical with (79) (*cf.* equations (53) and (74*a*)). Equation (81) now reduces to an identity, so that every configuration is formally a point of bifurcation. The interpretation is, of course, the same as that of § 11, the displacement for which $n = 1$ is a rigid body displacement, and so requires no force to maintain it. There is, of course, nothing of the nature of a change of stability, for v_n/C_n, instead of changing sign, remains permanently zero. The consideration of $n = 1$ is of no value except that it provides a check on the result of a rather involved series of analytical processes.

When there is homogeneity between core and crust the non-rotating system has been found to be stable for all displacements. To examine whether this is altered by the presence of the crust, it is natural to test first the extreme case in which the difference between the core and crust is as great as possible. Let us make the core so hot that its density is zero, so that κ has to be zero in order that the internal pressure may be maintained (*cf.* equation (12)).

Putting $\sigma = 0$, equations (79) and (80) reduce to

$$u_1(\kappa' a, \alpha) = 0, \qquad u_{n+1}(\kappa' a, \beta) = \infty, \quad . \quad . \quad . \quad (83)$$

or, by equation (74a),

$$J_{3/2}(\kappa' a, -\alpha) = 0, \qquad J_{n+1/2}(\kappa' a, \beta) = 0, \quad . \quad . \quad . \quad (84)$$

whence (equation (49)) α, β are given by

$$\tan\alpha = \frac{J_{3/2}(\kappa' a)}{J_{-3/2}(\kappa' a)}; \qquad \tan\beta = -\frac{J_{n+1/2}(\kappa' a)}{J_{-(n+1/2)}(\kappa' a)}. \quad . \quad . \quad . \quad (85)$$

The values of α, β corresponding to a few values of κa are given below—

$\kappa' a$.	α.	β.	
		$n = 2$.	$n = 3$.
	° ′	° ′	° ′
0	0	0	0
1	− 12 18	− 0 59	0 2
2	− 51 9	− 15 7	2 21
3	100 20	− 48 12	16 39
4	− 153 13	− 91 54	46 22
5	− 207 48	− 140 45	86 9
6	− 263 12	− 192 21	—
7	− 319 8	− 245 33	—
9	− 432 0	− 354 51	—
12	− 602 21	− 521 54	—

The case which is most favourable to the occurrence of points of bifurcation with positive values of ρ is when σ_0 falls to zero at the outer boundary. Let us accordingly examine this case. We have (equation (63))

$$J_{1/2}(\kappa' c, \alpha) = 0, \quad . \quad . \quad . \quad (86)$$

so that

$$\kappa' c = \pi - \alpha. \quad . \quad . \quad . \quad (87)$$

And in virtue of (86), the equation giving points of bifurcation (going back to equation (72)) is

$$v_n = \frac{4\pi c^{1/2}}{2n+1}\frac{Cn}{\kappa'}J_{n-1/2}(\kappa' c, -\beta), \qquad (88)$$

so that points of bifurcation of order n are given by

$$J_{n-1/2}(\pi-\alpha, -\beta) = 0. \qquad (89)$$

When $n = 2$, this becomes

$$\tan(\pi-\alpha+\beta) = \pi-\alpha;$$

when $n = 3$, it is

$$\tan(\pi-\alpha-\beta) = \frac{3(\pi-\alpha)}{3-(\pi-\alpha)^2}.$$

On treating these equations numerically it is found that they can never be satisfied We conclude that the non-rotating mass is stable for all displacements, subject, of course, to the condition that the density shall be everywhere positive.

Slow Rotation.

§ 23. We consider next the stability of a rotating mass of the type under consideration, in which we are limited to $\frac{w^2}{2\pi}$ being small compared with the density of the main mass. If we suppose that σ, the density of the core at its outer boundary, is equal to $\frac{w^2}{2\pi}$, we shall have a case—somewhat artificial of course—in which the density of the core is very small compared with that of the crust, and in which the equations are not too complex to admit of treatment.

We accordingly assume that $\sigma = \frac{w^2}{2\pi}$, and the equations (75), (76), and (77) (or (72)) reduce to the same equations as in the case of no rotation (equations (83)). Thus α, β have the same values as before, being given by the table on p. 479.

If we suppose that at the outer boundary of the crust the density falls to the small value $\sigma_0 = \frac{w^2}{2\pi}$, then the value of c, the radius of the outer boundary, is, as before, given by equations (86) or (87), and the value of v_n is still given by equation (88). Thus the analysis is exactly the same as in the case of no rotation, and there are no points of bifurcation.

It follows that, when σ_0 does not have this special value assigned to it, the only hope of finding points of bifurcation rests upon the gravitational tendency to instability which arises from the presence of the small layer of crust in which ρ has a value less than $\frac{w^2}{2\pi}$. Let us pass at once to the examination of the extreme case in which $\sigma_0 = 0$.

Denoting as before the density of the crust at its inner surface ($r = a$) by σ', and putting σ_0, the density at the outer surface ($r = c$) equal to zero, we have

$$C_0 a^{-1/2} J_{1/2}(\kappa' a, \alpha) = \sigma' - \frac{w^2}{2\pi},$$

$$C_0 c^{-1/2} J_{1/2}(\kappa' c, \alpha) = -\frac{w^2}{2\pi},$$

whence, on elimination of C_0,

$$\frac{2\pi\sigma'}{w^2} = 1 - \frac{c}{a}\frac{\sin(\kappa' a + \alpha)}{\sin(\kappa' c + \alpha)} \quad . \quad . \quad . \quad . \quad (90)$$

Equation (72) still gives

$$J_{n-1/2}(\kappa' c, -\beta) = 0,$$

as the condition for points of bifurcation, and when $n = 2$ (the only case which appears to be worth examining), this reduces to

$$\tan(\kappa' c + \beta) = \kappa' c, \quad . \quad . \quad . \quad . \quad (91)$$

in which β is given from the table on p. 479. The procedure is to find $\kappa' c$ from equation (91), and hence calculate the value of $w^2/2\pi\sigma'$ from equation (90). The results for a few values of $\kappa' a$ are given in the table following (the last column is explained later) :—

$\kappa' a$.	$\kappa' c$.	c/a.	$w^2/2\pi\sigma'$.	$w^2/2\pi\theta$.
	° ′			
0	4·489 = 257 27	∞	∞	0·40
1	4·475 = 256 54	4·475	0·2221	0·45
2	4·713 = 269 21	2·356	0·2269	0·50
3	5·380 = 308 16	1·793	0·2159	0·52
4	6·153 = 352 37	1·538	0·1821	0·51
5	7·026 = 402 40	1·405	0·1570	0·50
6	7·944 = 455 7	1·324	0·1365	0·49
7	8·886 = 509 9	1·269	0·1198	0·48

The obvious remark must at once be made that probably all the values for $w^2/2\pi\sigma'$ are too large for results obtained by the neglect of w^4 to be accurate. But apart from absolute accuracy there is an obvious tendency for the value of $w^2/2\pi\sigma'$ to fall off as $\kappa' a$ increases—for lower values of c/a the symmetrical configuration becomes unstable for lower and lower values of $w^2/2\pi\sigma'$. For $\kappa' a = 100$, the value of $w^2/2\pi\sigma'$ is 0·0104.

§ 24. Against this, it must be noticed that the value of $w^2/2\pi\sigma'$ is of very slight importance; what we are concerned with is the ratio of $w^2/2\pi$ to the mean density of the whole mass. For a very rough calculation, we may assume the mean density of the crust to be $\frac{1}{2}\sigma'$, whence it follows that the mean density of the whole mass will be roughly equal to a density θ defined by

$$\theta = \tfrac{1}{2}\sigma' \frac{c^3 - a^3}{c^3},$$

and the value of $w^2/2\pi\theta$ will be approximately the same as the quantity $w^2/2\pi\rho$ which is computed from observations of binary stars. Values of $w^2/2\pi\theta$ are given in the last column of the table on the preceding page; the value 0·40 corresponding to $\kappa' a = 0$ (no core) being inserted from the result of the previous analysis (§ 13). As before, the numbers are not numerically accurate, but their general trend may be expected to reveal the general trend of the true series of numbers. It at once appears that the values of $w^2/2\pi\theta$ are surprisingly steady: there is certainly no rapid decrease in their amount as $\kappa' a$ increases.

Summary and Conclusion.

§ 25. The problem we have had under consideration has been that of testing whether the behaviour of a rotating mass of compressible heterogeneous matter differs very widely from that of the incompressible homogeneous mass which has been studied by MACLAURIN, JACOBI, POINCARÉ, and DARWIN. The result obtained can be summed up very briefly by saying that the ideal incompressible mass has been found to supply a surprisingly good model by which to study the behaviour of the more complicated systems found in astronomy. The problem especially under consideration has been that of determining the amount of rotation at which configurations of revolution (*e.g.*, spheroids) first become unstable. In so far as we have been able to examine the question, it appears probable that the compressible mass will behave, at least up to this point, in a manner almost exactly similar to the simpler incompressible mass, and results obtained for the latter will be nearly true, both qualitatively and quantitatively, for the former. The compressible mass, set into rotation, will apparently pass through a series of flattened configurations very similar to the Maclaurin spheroids; it will then, for just about the same amount of rotation (as measured by $w^2/\bar{\rho}$), leave the symmetrical form and assume a form similar to the Jacobian ellipsoids. Beyond this stage our analysis has not been able to deal with the problem. Indeed, strictly speaking, our analysis has hardly been able to carry this far. A question of importance has been whether the quasi-spheroidal form for a compressible mass does not become unstable for a much smaller value of w^2 than the incompressible mass, and whether the instability does not set in in a different way. These questions we have been able to answer, with, I think, a very high degree of probability, in the negative. The whole matter is of necessity one of probability only, and not of certainty, for the general heterogeneous compressible mass is not amenable to analysis until a great number of simplifying assumptions have been made.

It was first pointed out that a compressible mass has an infinitely greater number of vibrations than an incompressible one, and as the mass is only stable when every vibration individually is stable, it might be thought that a compressible mass had more chance of being unstable—or would become unstable sooner—than the corresponding compressible mass. This has on the whole been found not to be the case, and on looking through the analysis the reason can be seen.

A vibration in a compressible mass may be regarded loosely as a system of waves; the distance from one point of zero displacement to the next may be regarded as a sort of wave-length of the vibration. The stability or instability of a vibration depends on which is the greater—the gain in elastic energy or the loss in gravitational energy when the vibration takes place. But as between a vibration of great wave-length and one of short wave-length there is this important distinction: for equal maximum amplitudes the gravitational disturbance caused by the disturbance of great wave-length is much greater than that caused by the disturbance of short wave-length, since the elements of the latter very largely neutralise one another. Thus the change in gravitational energy is enormously the greatest for disturbances of great wave-length, while it is easily seen that the changes in elastic energy are approximately the same. It follows that if the mass becomes unstable it will be through a vibration of the greatest possible wave-length, *i.e.*, a wave-length about equal to the diameter of the mass. This general prediction is amply verified in the detailed problems that have been discussed. When we reflect that the vibrations of greatest wave-length are exactly those which are common both to compressible and incompressible masses, we see readily that, in this respect at least, compressibility is likely to make but little difference.

The vibrations of greatest wave-length are put in evidence, both in the compressible and incompressible mass, by the displacement of the surface. A vibration in which the displacement is proportional to a zonal harmonic P_n may be thought of as having a wave-length approximately equal to $\pi a/n$. In accordance with the principle that vibrations of great wave-lengths are most effective towards instability, we should expect the lowest values of n to give the vibrations which first become unstable, and this is, in fact, found to be the case. But here a very real distinction enters between the compressible and the incompressible mass. In the incompressible mass vibrations of order $n = 1$ are non-existent, the displacement being purely a rigid body displacement; in the compressible mass vibrations of order $n = 1$ can certainly occur, and so might reasonably be expected to be the first to become unstable.

It is in point of fact known that the incompressible mass becomes unstable through vibrations of orders 2, 3, ... in turn; it is found that the compressible mass also becomes unstable through vibrations of orders 2, 3, ... in turn, the vibrations of order 1 failing completely to produce instability. The reason for this apparent anomaly can, I think, be traced in the following way. In a displacement of order 1 any spherical layer of particles will after displacement be spread uniformly over another sphere excentric to the first. The gravitational force produced by this sphere of particles both before and after displacement is exactly *nil* at a point inside the sphere. Thus the gravitational field set up by a displacement of order 1 neutralises itself in a way not contemplated in the general argument outlined above. Also the vibrations of order 1 and of greatest wave-length in the interior are not available, for they represent solely a rigid body displacement.

The question of vibrations of order 1 is treated in §§ 3–7; it is shown that they may be disregarded, and we pass to the consideration of vibrations of orders 2, 3, ..., expecting (as, in fact, is found to be the case) that instability will first set in through a vibration of order 2.

It is only possible to discuss special cases, and the one which is most amenable to analysis is that in which the pressure and density are connected by LAPLACE's law, $p = c(\rho^2 - \sigma^2)$. It is first proved (§§ 8–11) that, for a mass of such matter at rest, the spherical form is stable for all displacements. Later (§§ 15–22) it is shown that this is true when c varies inside the mass; it is true even up to the case which is the most likely to be unstable, in which the matter in the interior is of negligible density and the main part of the mass is collected in a surface crust—a sort of astronomical soap-bubble.

We proceed next to examine for what amount of rotation these figures will become unstable, treating first the case in which c is the same throughout the mass. Imagining c and σ to vary we can get a variety of types of mass. The surprising result is obtained (by something short of strict mathematical proof) that the figure which is the first to become unstable (as $w^2/2\pi\bar{\rho}$ increases uniformly for them all) is the perfectly incompressible one—gravitational instability appears to act in the unexpected direction, at any rate when the degree of rotation is measured by $w^2/2\pi\bar{\rho}$, $\bar{\rho}$ being the mean density. As it was not possible to obtain strictly accurate figures in this case, the result was checked by considering the artificial, but physically analogous, problem of rotating cylinders of Laplacian matter, in which it was possible to obtain perfectly exact results (§ 14). The result was confirmed, and the additional information was obtained that the value of $w^2/2\pi\bar{\rho}$ remains surprisingly steady through quite a wide range of compressibility (*vide* table on p. 471).

The physical reason for this can, I think, be understood as follows. The more compressible the matter is the more it tends to concentrate near the centre, *i.e.*, in just those regions where the "centrifugal force" obtains, so to speak, least grip on it. Incompressibility neutralises the gravitational tendency to instability, but tends to compel the matter to place itself so that the rotational tendency to instability can act at the best advantage.

The similar problem is next investigated (§§ 23, 24) when c varies inside the mass; in particular, the limiting case of a soap-bubble-like mass is considered. Again the surprising result emerges that the value of $w^2/2\pi\bar{\rho}$ needed to establish instability of the symmetrical configuration is just about the same as before (*vide* table, p. 481). The matter is now constrained to remain, so to speak, on the rim of a fly-wheel where the centrifugal force can act at the best advantage and gravitational instability has full scope. If ρ is the mean density of the crust, $w^2/2\pi\rho$ must obviously be less than before. But if $\bar{\rho}$ is the mean density of the whole mass, $\rho/\bar{\rho}$ is also much smaller. These two quantities march with approximately equal steps, so that $w^2/2\pi\bar{\rho}$ remains almost unaltered.

Thus we have the general result that for all the varied types of mass that have been considered the spheroidal or quasi-spheroidal form always becomes first unstable for just about the same value of $w^2/2\pi\bar{\rho}$. If, from the point of view of discovering new processes in nature, the present investigation has been somewhat barren, at least we may reflect that the work of DARWIN and POINCARÉ has been shown, to some extent, to have an enhanced value, in that it seems to apply to the real bodies of nature and not merely to mathematical abstractions.